Lecture 24 04/24/2019

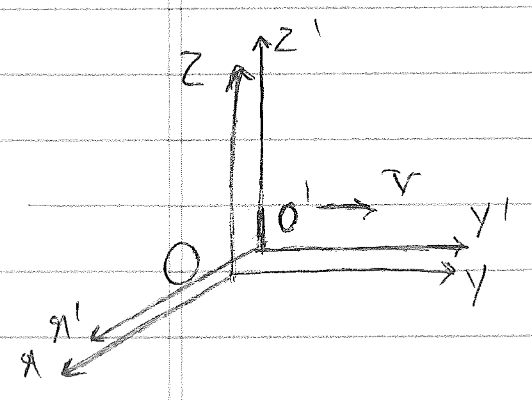
Special Relativity

Special theory of relativity is based on two postulates:

(1) All laws of nature have the same form (i.e., are "covariant") in inertial reference frames.

(2) Speed of light has the same value "c" in all inertial reference frames.

Based on these postulates, one can obtain the relation between the spatial and time cordinates of a point in two inertial frame, called "Lorentz transformations". For example, consider two reference frames with relative motion in the x direction as follows:



If O and O' coincide at $t=t'=0$, we have:

$$\begin{cases} x' = \dfrac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} \\ t' = \dfrac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \\ y' = y \\ z' = z \end{cases}$$

Lorentz transformations

A fundamental conceptual difference from Newtonian physics is that time is no longer absolute. I.E., as we change frames time also changes the same as spatial coordinates do. Therefore, in special relativity we have "spacetime" as opposed to space + time in Newtonian physics.

The coordinates of a point in spacetime form a 4-vector:

$$x_\nu \equiv (ct, x, y, z) = (ct, \vec{x})$$

Considering two nearby points in the spacetime, it can be shown that the following quantity is the same in all frames:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

This leads us to define the distance between two nearby points in the spacetime as follows:

$$ds^2 = \eta^{\nu\nu} dx_\nu dx_\nu$$

Where $\eta^{\nu\nu}$ is the metric that has the following form in all frames:

$$\eta^{\nu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\eta_{\nu\nu}$ is defined as $(\eta^{\nu\nu})^{-1}$, as a matrix, which is the same as $\eta^{\nu\nu}$.

One can use the metric to move the indice up an down:

$$x^\nu \equiv \eta^{\nu\nu} x_\nu \quad , \quad x_\nu \equiv \eta_{\nu\nu} x^\nu$$

x_ν and x^ν are the "covariant" and "contravariant", respectively, components of the same 4-vector. We note that repeated indices (once as a subscript and once as a superscript) imply summation according to the

Einstein summation convention.

In general, quantities A_ν (A^ν) that have the same properties under Lorentz transformation as x_ν (x^ν) form covariant (contravariant) components of a 4 vector:

$$A'_\nu = \frac{\partial x'_\nu}{\partial x_N} A_N = \frac{\partial x^N}{\partial x'^\nu} A_N$$

$$A'^\nu = \frac{\partial x'^\nu}{\partial x^N} A^N$$

Note that $\frac{\partial x'_\nu}{\partial x_N}$ are matrix elements of a Lorentz transformation.

For example for a boost in x direction (discussed above),

$$\Lambda^N{}_\nu = \frac{\partial x'_\nu}{\partial x_N}$$

$$\Lambda = \begin{bmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where:

$$\eta = \tanh^{-1}\beta \,, \quad \beta \equiv \frac{v}{c}$$

For a rotation by angle θ about the z axis, we have:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A very important example of a 4-vector is the 4-momentum of a particle with mass m:

$$P_\nu = (\gamma m, \gamma m \vec{v}) \quad , \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

An important example of a 4-vector in electrodynamics in the 4-current defined as:

$$J_\nu = (c\rho, \vec{J})$$

The continuity equation $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$, which implies conservation of the electric charge, can the be written as $\partial^\nu J_\nu = 0$.

Another important example is the 4-vector A_ν that brings the scalar and vector potentials together.

$$A_\nu = (\Phi, c\vec{A})$$

The condition for Lorenz gauge $\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t}$ can now be written as:

$\partial^\nu A_\nu = 0$

The wave equations for Φ and $\vec{A}$ in the Lorenz gauge can be written in covariant form in terms of A_ν and J_ν:

$$\left.\begin{aligned}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} &= -\mu_0 \vec{J} \\ \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi &= -\frac{\rho}{\varepsilon_0}\end{aligned}\right\} \Rightarrow \boxed{\partial^\nu \partial_\nu A_N = -\frac{1}{c\varepsilon_0} J_N}$$

The inner product of two 4-vectors A_ν and B_ν is defined as:

$$A \cdot B = A^\nu B_\nu = A_\nu B^\nu$$

It is straightforward to show that $A \cdot B$ is Lorentz invariant.

A rank-2 tensor is an object whose covariant and contravariant components have the following properties under Lorentz transformations:

$$F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma} F^{\rho\sigma}$$

$$F'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} F_{\rho\sigma}$$

Generalization to higher rank tensors and mixed tensors is straightforward. The indices on tensors can also be raised and lowered by using a (string of) $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ respectively.

An important example of a tensor in electrodynamics is the "field strength" tensor:

$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \qquad (F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu)$

This is an anti-symmetric rank-2 tensor whose contravariant components form a 4x4 matrix as follows:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix}$$

The tensor nature of $F^{\mu\nu}$ ($F_{\mu\nu}$) can be shown by using the fact that A^μ (A_μ) is a 4-vector.

The "dual field strength" tensor is defined as:

$F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ ($\varepsilon^{\mu\nu\rho\sigma}$: Ricci-Levi Civita anti-symmetric tensor)

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & E_y & 0 \end{bmatrix}$$

The homogeneous Maxwell equations $\vec{\nabla}.\vec{B}=0$ and $\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}$

can now be written in covariant form:

$$\boxed{\partial_\nu \tilde{F}^{\nu\mu}=0}$$

The inhomogeneous Maxwell equations $\vec{\nabla}.\vec{E}=\frac{\rho}{\varepsilon_0}$ and $\vec{\nabla}\times\vec{B}-\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}=\mu_0\vec{J}$

can also be written in covariant form:

$$\boxed{\partial_\nu F^{\nu\mu}=\frac{J^\mu}{c\varepsilon_0}}$$

We can construct two Lorentz-invariant quantities from F and $\tilde{F}$:

$F^{\nu\mu}F_{\nu\mu}=2(c^2|\vec{B}|^2-|\vec{E}|^2)$, $\tilde{F}^{\nu\mu}F_{\nu\mu}=-4c\vec{E}.\vec{B}$

The fact that they are Lorenz invariant implies the following:

(1) A purely $\vec{E}$ field in one frame, cannot be purely $\vec{B}$ in any other frame.

(2) If $\vec{E}\perp\vec{B}$ in one frame, the $\vec{E}\perp\vec{B}$ in all frames.

knowing how $F^{\nu\mu}$, $F_{\nu\mu}$ behave under Lorentz transformations, one can find the transformation laws for $\vec{E}$ and $\vec{B}$ fields:

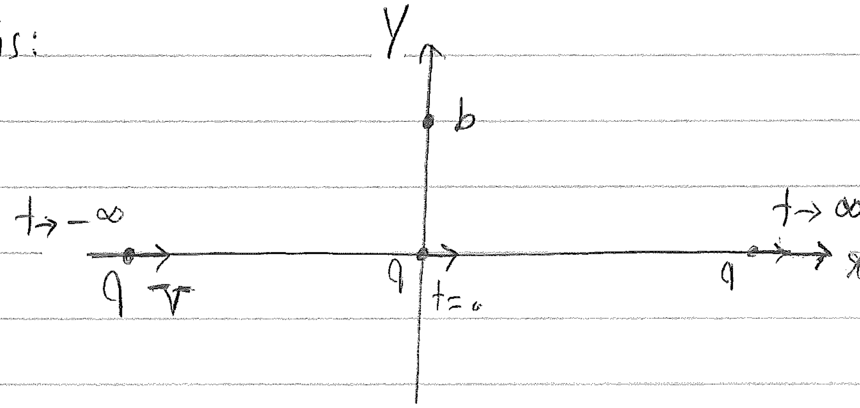
$$\begin{cases} \vec{E}' = \gamma(\vec{E} + \vec{\beta} \times c\vec{B}) - \dfrac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}) \\ \vec{B}' = \gamma(\vec{B} - \dfrac{1}{c}\vec{\beta} \times \vec{E}) - \dfrac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B}) \end{cases}$$

We see that a boost generally mixes the electric and magnetic fields. Considering a boost in the x direction, we have:

$$\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - \beta c B_z) \\ E'_z = \gamma(E_z + \beta c B_y) \end{cases} \qquad \begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y + \dfrac{\beta}{c} E_z) \\ B'_z = \gamma(B_z - \dfrac{\beta}{c} E_y) \end{cases}$$

Example: The electromagnetic field of a uniformly moving point charge.

Consider a point charge q moving with constant velocity in the x direction along the x axis:



We want to find the $\vec{E}$ and $\vec{B}$ fields at the point $(0, b)$ as a function of time.

The space time coordinates of the observation point are

$(ct, 0, b, 0)$. It will be easier to move to the rest frame of q where there is only a Coulomb electric field, and then transform back to the original frame. Note that:

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} \underset{x=0 \text{ for the observation point}}{=} -\gamma vt \quad , \quad y' = y = b$$

In the rest frame of q, we have:

$$B'_x = B'_y = B'_z = 0$$

$$E'_x = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^2}\frac{x'}{r'} \quad , \quad E'_y = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^2}\frac{y'}{r'} \quad , \quad E'_z = 0$$

Here, $r' = \sqrt{x'^2 + y'^2}$, where $x' = -\gamma vt$ and $y' = b$. Thus:

$$E'_x = -\frac{q}{4\pi\varepsilon_0}\frac{\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \quad , \quad E'_y = \frac{q}{4\pi\varepsilon_0}\frac{b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

This results in:

$$\begin{cases} E_x = -\dfrac{q}{4\pi\varepsilon_0}\dfrac{\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \\[2ex] E_y = \dfrac{q}{4\pi\varepsilon_0}\dfrac{\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \\[2ex] B_z = \beta E_y \end{cases}$$

(All other fields vanish)

We can find this result using the Lienard-Wiechert potential (which we will see later). However, that will considerably more complicated than using the transformation laws for $\vec{E}$ and $\vec{B}$.

As $|\beta| \to 1$, the electric field lines will be concentrated around the y axis (as opposed to being isotropic in the rest frame of the charge). This is in agreement with the length contraction in special relativity.

Finally, let us point out that the instantaneous rest frame of an accelerating point charge is suitable to find the radiated power. Power in the original frame $\frac{dE}{dt}$ is equal to that in the instantaneous rest frame $\frac{dE'}{dt'}$ since both of dE and dt are time components of 4-vectors, and hence their ratio is a Lorentz-invariant quantity. However, it is often easier to calculate the power in the instantaneous rest frame of the charge.